FINAL: ALGEBRA II

Date: **5th May 2019**
The Total points is **110** and the maximum you can score is **100** points.

All vector spaces considered below are assumed to be finite dimensional.

(1) (6+6+6+7=25 points) Answer the following multiple choice questions about each of them. Write all correct options. No justification needed. **No partial credit will be given if a correct option is missing or an incorrect option is written.**

(a) Let A and B be two matrices. Which of the following statements are true?
   (i) $\det(AB) = \det(A)\det(B)$
   (ii) $\det(-A) = -\det(A)$
   (iii) $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$
   (iv) $\mathrm{tr}(-A) = -\mathrm{tr}(A)$

(b) Let A and B be two distinct bases of a vector space V. Which of the following statements are true?
   (i) $A \cup B$ is a generating set.
   (ii) $A \cup B$ is a linearly independent set.
   (iii) A and B have same number of elements.
   (iv) If $C \subset A \cup B$ then C is either a linearly independent set or a generating set.

(c) Let $\phi : V \to W$ be a linear map of vector spaces. Let A be the matrix of ϕ with respect to the bases $(\mathscr{B}, \mathscr{C})$ of V and W respectively. Which of the following are true?
   (i) $\mathrm{rank}(A) \leq \dim(\ker(\phi))$
   (ii) $\mathrm{rank}(A) \leq \dim(\mathrm{im}(\phi))$
   (iii) $\mathrm{rank}(A) \geq \dim(\ker(\phi))$
   (iv) $\mathrm{rank}(A) \geq \dim(\mathrm{im}(\phi))$

(d) Let V and W be vector spaces. Let ϕ and ψ be linear operators V and W respectively. Consider the linear operator θ on $V \oplus W$ given by $\theta(v, w) = (\phi(v), \psi(w))$ for $v \in V$ and $w \in W$. Which of the following are true?
   (i) $\det(\theta) = \det(\phi)\det(\psi)$
   (ii) $\mathrm{tr}(\theta) = \mathrm{tr}(\phi) + \mathrm{tr}(\psi)$
   (iii) The minimal polynomial of θ is the product of the minimal polynomials of ϕ and ψ.
   (iv) The characteristic polynomial of θ is the product of the characteristic polynomials of ϕ and ψ.

(2) (8+8+8+8=32 points) Prove or disprove using a counterexample the following statements.
  (a) Every real square matrix is conjugate to a real upper-triangular matrix.
  (b) Let $x(t)$ be a column vector of functions of length n and A be a $n \times n$ complex matrix. The differential equation $\frac{d}{dt}x(t) = Ax$ has a solution.
  (c) Let $(V, \langle \cdot, \cdot \rangle)$ be a vector space together with a symmetric bilinear form. Let A be the matrix of the bilinear form with respect to some basis. If $\det(A) = 1$ then A is positive definite.
  (d) Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be two vectors in $\mathbb{C}^2$. The function $\langle X, Y \rangle = x_1y_1 + ix_1y_2 - ix_2y_2 + ix_2y_2$ from $\mathbb{C}^2$ to $\mathbb{C}$ is a hermitian form.

(3) (3+10=13 points) Define orthogonal matrix. Let A be a real orthogonal matrix of determinant -1. Show that -1 is an eigenvalue of A.

(4) (3+17=20 points) Let V be a vector space of dimension n over a field k. Let T be a linear operator on V. Define minimal polynomial of T. Show that if $T^m = 0$ for some $m \geq 0$ then $T^n = 0$.

(5) (5+15=20 points) Define normal operator and self-adjoint operator on a finite dimensional hermitian space V. Let T be a normal operator on V. Show that T is self-adjoint iff all the eigenvalues of T are real.